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A remark on ∞-distributed sequences

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1. Introduction

The theory of asymptotic distribution modulo one had been investigated from various interests, the behaviour of the fractional parts of linear forms with integer variables as roots, on irrational numbers by Sierpinski, on probability by Borel and F. Bernstein and on diophantine approximation and Fourier series by Hardy-Littlewood. These ideas were concentrated in Weyl's work of 1916, where by a simple definition, he coined the general notion of uniform distribution or equidistribution (mod 1) of a sequence of real numbers and as the first gave a necessary and sufficient condition.

Weyl's treatment of this problem opened the way for numerous metrical investigations. His method had been refined by Menchoff, Plancherel, Rademacher, Erdös, a.o., and become the source of many results in the large domain where measure theory, probability theory and ergodic theory meet with number theory.

After this tribute to Weyl's work, his simple definition was slightly modified or generalized from various point of view. A modification is that of complete distribution (mod 1) due to Koroboff[7] in 1949. Afterwards, Franklin[3] gave the definition of completely equidistributed sequence in 1963 independently and Knuth[10] gave an example of completely equidistributed sequence and Haber's work[4] was a slight modification. But these two definitions are the same and Koroboff's subsequent works[8][9] and Starčenko[12] contained more general results.

In this paper, above definitions and results will be explained and several new results will be added from the notion of linear dependency. Their details and proofs will be remained and some further new results will be discussed in another journal.

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2. Definition and Weyl's criterion

Firstly, we give the definition. Consider a sequence (x_n) :

(1) $$x_1, x_2, \ldots$$

of real numbers and their fractional parts (residues mod 1)

$$\{x_1\}, \{x_2\}, \ldots, \quad \text{i.e.} \quad x_1-[x_1], x_2-[x_2], \ldots$$

which are all contained in the unit interval $I_0 : 0 \leq x < 1$, and take a fixed interval $I : a \leq x < b$; $I \subset I_0$.

Let $N_n(I)$ be the number of x_j among the first n numbers which are situated in I. Then if and only if for each fixed $I \subset I_o$, $N_n(I)/n \to b-a$ as $n \to \infty$, the sequence (1) will be called uniformly distributed (mod 1) (equidistributed mod 1 ; gleichverteilt mod 1 ; équiréparti mod 1). The following theorems are well known and the first one relates to ergodic theory and the other is Weyl's criterion .

Theorem 1. The necessary and sufficient condition that a sequence (x_n) is uniformly distributed mod 1 is that for any Riemann-integrable function on I_0,

$$\sum_{j=1}^{n} f(\{x_j\})/n \longrightarrow \int_0^1 f(x)\,dx \ .$$

Theorem 2. (Weyl's criterion) The necessary and sufficient condition that a sequence (x_n) is uniformly distributed mod 1 is that for $m = \pm 1, \ \pm 2, \ldots,$

$$\sum_{j=1}^{n} e^{2\pi i m x_j} = o(n)$$

An n-dimensional analogue of this definition and criterion leads us to the notion of complete distribution. Take a sequence $(z^{(i)})$ of n-dimensional vectors of real numbers

(2) $z^{(1)} = (y_1^{(1)}, y_2^{(1)}, \ldots, y_n^{(1)}), \ z^{(2)} = (y_1^{(2)}, y_2^{(2)}, \ldots, y_n^{(2)}), \ldots,$

and their fractional parts are denoted by $\bar{z}^{(i)}$ which all fall in n-dimensional unit cube $I_0^{(n)} = \overbrace{I_0 \times \cdots \times I_0}^{n}$ and take a fixed interval $I \subset I_0^{(n)}$. Then if and only if for each fixed $I \subset I_0^{(n)}$, $N_n(I)/n \to \mu(I)$ as $n \to \infty$ where $\mu(\cdot)$ is the Lebesgue measure, the sequence (2) will be called uniformly distributed (mod 1) by n's. Then n-dimensional corresponding Weyl's criterion is the following.

Theorem 3. The necessary and sufficient condition that a sequence $(z^{(i)})$ is uniformly distributed mod 1 by n's is that for each fixed n-tuple of integers $(m_1, \ldots, m_n)$ which are not equal simultaneously to zero,

$$\sum_{j=1}^{n} e^{2\pi i(m_1 y_1^{(j)} + \cdots + m_n y_n^{(j)})} = o(n) .$$

Now for a given sequence (x_n) the derived k-dimensional sequence is defined by

$$z_k^{(1)}=(x_1, x_2, \ldots, x_k),\ z_k^{(2)}=(x_2, \ldots, x_{k+1}), \ldots, z_k^{(j)}=(x_j, \ldots, x_{j+k-1}), \ldots .$$

A sequence (x_n) is completely uniformly distributed (mod 1), (or ∞ -distributed sequence for abbreviation), if for all k the derived k-dimensional sequence is uniformly distributed by k's .

Koroboff's definition was stated with the help of Weyl's criterion, which would lead the misunderstanding by Franklin[3] and Knuth [10] [11] (I suppose).

3. Koroboff's results and relations to other results

The first problem is to determine the class of functions with uniformly distributed fractional parts and with completely uniformly distributed fractional parts. A function f(x) is uniformly distributed or completely uniformly distributed if the sequence

$$f(1),\ f(2),\ \ldots,\ f(n),\ \ldots$$

generated by $f(\cdot)$ is uniformly distributed mod 1 or completely uniformly distributed (mod 1) respectively. A basic result was due to Fejér .

Theorem 4. (Fejér) Let $g(t) > 0$ be a continuous increasing function with a continuous derivative $g'(t)$ for $1 \leq t < \infty$ and satisfying the following conditions :

(i) $g(t) \to \infty$, as $t \to \infty$,

(ii) $g'(t) \to 0$ monotonically, as $t \to \infty$,

(iii) $t g'(t) \to \infty$, as $t \to \infty$.

Then the function $g(x)$ is uniformly distributed.

But Koroboff gave a new class of functions with uniformly distributed fractional parts.

<u>Theorem 5.</u> (Koroboff) Let a function $f(x)$ be defined by the series :

$$f(x) = \sum_{k=0}^{\infty} a_k x^k , \quad |a_k| = e^{-\omega(k)} .$$

If for all sufficiently large k, the following conditions :

(i) $\omega(k) \geqslant k^{\lambda}$ for some constant $\lambda > 3$,

(ii) $(1 + 1/k)\omega(k) \leqslant (k + 1) \leqslant k\,\omega(k)$,

are satisfied, then the function $f(x)$ is uniformly distributed.

<u>Corollary</u>. Let a function $f(x)$ be defined in the theorem. If for all sufficiently large k, the following conditions :

(i) $\omega(k) \geqslant k^{\lambda}$ for some constant $\lambda > 3$,

(ii) $(1 + \beta_1/k)\omega(k) \leqslant \omega(k + 1) \leqslant \beta_2 k\,\omega(k)$

for some constants $\beta_1 > 1$ and $\beta_2 < 1$,

are satisfied, then the function $f(x)$ is completely uniformly distributed.

<u>Remark.</u> Let $f(x)$ be the function considered in the theorem.

Then the function f(x) increases faster than $x^{(\log x)^{(1/2)-\varepsilon}}$ for any $\varepsilon > 0$, but $f(x) = o(x^{(\log x)^{1/2}})$.

Indeed, if we select the function of the form

$$f(x) = \sum_{k=0}^{\infty} e^{-k^{\lambda}} x^k, \quad \lambda = 1 + 2/(1-\varepsilon) \qquad (0 < \varepsilon < 1),$$

and take

$$k = k_2 = [(\log x/\lambda)^{1/(\lambda-1)}],$$

then

$$f(x) > e^{-k_2^{\lambda} + k_2 \log x} > e^{c_1(\lambda)(\log x)^{1+1/(\lambda-1)}} > x^{(\log x)^{(1/2)-\varepsilon}}$$

Then $f'(x) \to \infty$, as $x \to \infty$ (if $f'(x)$ exists) i.e. f(x) does not satisfy the condition (ii) in the theorem 4.

As a consequence of theorem 5, we get the following theorem.

Theorem 6. Let a function f(x) be defined by the everywhere convergent series :

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad |a_k| = e^{-\omega(k)}.$$

If there exist constants $G > g > 2$ such that

$$g\ \omega(k) < \omega(k+1) < G\omega(k)$$

for all sufficiently large integer k, then the function f(x) is completely uniformly distributed.

Remark. The growth of function f(x) considered in theorem 6 satisfies the following conditions :

$$f(x) = o(x^{\lambda_1 \log\log x}), \quad \text{for all } \lambda_1 > 1/\log g ,$$

and

$$f(x) = \Omega(x^{\lambda_2 \log\log x}), \quad \text{for all } \lambda_2 < 1/\log G .$$

An example of a completely uniformly distributed sequence was given by Starčenko. Put $n_k = [1+\exp(k^3)]$ ($k=1, 2, \ldots$) ; also denote by $p_1=2$, $p_2=3, \ldots$ the successive primes. Then the sequence

$\{\log 2\}, \{2\log 2\}, \ldots, \{n_1 \log 2\}$;

$\{\log 2\}, \{\log 3\}, \{2\log 2\}, \ldots, \{n_2\log 2\}, \{n_2 \log 3\}$;

$\ldots$;

$\{\log p_1\}, \{\log p_2\}, \ldots, \{\log p_r\}, \{2\log p_1\}, \{2\log p_2\}, \ldots, \{2\log p_r\}, \ldots,$

$$\{n_r \log p_1\}, \{n_r \log p_2\}, \ldots, \{n_r \log p_r\};\ \ldots,$$

is completely uniformly distributed.

Another example was given by Knuth, which was constructed only by dyadic rational numbers.

The definition that a function $f(x)$ is completely uniformly distributed is also stated as the following : for each fixed integers $s \geqslant 1$, $a_1, \ldots, a_s \neq 0, \ldots, 0$, the sequence of numbers

$$g(n) = a_1 f(n) + \cdots + a_s f(n+s-1) \qquad (n=1, 2, \ldots)$$

is uniformly distributed (mod 1). For $s=h$, $a_1=-1$, $a_s=+1$, $a_j=0$ $(2 \leqslant j \leqslant s-1)$, we have the special sequence considered in van der Corput's difference theorem.

<u>Theorem 7.</u> (van der Corput) Let $g_h(x) = g(x+h) - g(x)$ $(h=1, 2, \ldots)$. If the function $g_h(x)$ is uniformly distributed for any h, then the function $g(x)$ is uniformly distributed.

The inverse of this theorem is not true, for $g(x) = \alpha x$ where α is an irrational number. Put $g(x) = \alpha x^2$. Then the function $g(x)$ satisfies the condition of theorem 7, but $g(x)$ is not uniformly distributed by $\mathfrak{z}$'s.

4. From the scope of linear dependency

If we consider random sequences, a completely uniformly distributed sequence is supposed to have two properties of randomness, uniformity (identically distributed) and independency. But the property of independency is too strong condition, then correlation coefficients (serial correlation coefficients, in this case) are used as a weaker condition of linear independency from some statistical view points.

For each $h = 1, 2, \ldots,$ a function $\tau_h(\omega)$ is defined for a sequence ω, which denotes the sequence (1), in the following way :

$$\tau_h(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\{x_{j+h}\} - 1/2)(\{x_j\} - 1/2),$$

if the indicated limit exists.

A sequence ω is called white if $\tau_h(\omega)=0$ for any h.

Then every completely uniformly distributed sequence is white. More precisely, if for any $h=1, 2, \ldots$ the derived two-dimensional sequence

$$(x_1, x_h),\ (x_2, x_{h+1}), \ldots$$

is uniformly distributed mod 1 by 2's, then the sequence is white.

The function αx, where α is an irrational number, is uniformly distributed, but not white. But the function αx^2 satisfying the condition of van der Corput difference theorem is white, but not uniformly distributed by 3's and not white of order three.

The function x^{σ} $(0 < \sigma < 1)$ satisfying the conditions of Fejér's theorem is not white.

But the notion of uniformity differs from that of independency. Since the function $1/x$ is not uniformly distributed but white (of course,the function $\tau_h(\cdot)$ is slightly modified but well-defined).

The function $\tau_h(\cdot)$ was named autocorrelation function by Jagerman[5] which is not the same as in the works of Cigler[2] and Bertrandias[1].

These further results or details will be discussed in another journal.

References

[1] Bertrandias, J. P. (1964), Suites pseudo aleatoire et criteres d'équirépartition modulo un, Comp. Math., 16, 23-28.

[2] Cigler, J. (1964), The fundamental theorem of van der Corput on uniform distribution and its generalizations, Comp. Math., 16, 29-34.

[3] Franklin, J. N. (1963), Deterministic simulation of random processes, Math. Comp., 17, 28-57.

[4] Haber, S. (1970), Sequences of numbers that are approximately completely equidistributed, J. of A C M, 17, 269-272.

[5] Jagerman, D. L. (1963), The autocorrelation function of a sequence uniformly distributed modulo 1, Ann. Math. Stat., 34, 1243-1252.

[6] Koksma, J. F. (1964), The theory of asymptotic distribution modulo one (General introduction), Comp. Math., 16, 1-22.

[7] Koroboff, N. M. (1949), Einige Probleme der Verteilung von Bruchteilen (in russ.), Uspechi Mat. Nauk, 4, 189-190.

[8] Koroboff, N. M. (1950), On some problems of uniform distribution (in russ.), Izv. Akad. Nauk SSSR, 14, 215-238.

[9] Koroboff, N. M. (1956), On completely uniformly distributed and jointly normal numbers (in russ.), Izv. Akad. Nauk SSSR, 20, 649-669.

[10] Knuth, D. E. (1965), Construction of a random sequence, BIT, 5, 246-250.

[11] Knuth, D. E. (1969), The art of computer programming, 2, Seminumerical Algorithms, Addison-Wesley Publishing Co. .

[12] Starčenko, L. P. (1959), The construction of completely uniformly distributed sequence (in russ.), Dokladi, 129, No. 3, 519-521.

[13] Weyl, H. (1916), Über die Gleichverteilung von Zahlen mod Eins, Math. Ann., 77, 313-352.